Horn clauses

• A **literal** is an atomic formula or its negation
• A **clause** is a disjunction of literals
• A **Horn clause** is a clause with exactly one positive literal
• A **Horn formula** is a conjunctive normal form formula whose clauses are all Horn
Example

• Prolog:

\[
c:\leftarrow a, b.
a.
b.
\]

• Horn formula:

\[
[c \lor \neg a \lor \neg b] \land a \land b
\]

\[
[c, \neg a, \neg b] \quad [a] \quad [b]
\]
• Horn formula: $[c, \neg a, \neg b] \ [a] \ [b]$

• Let us attempt to prove $c$ by contradiction

• Hence, the goal clause is $\neg c$

$[c, \neg a, \neg b] \ [a] \ [b] \ [\neg c]$

• By resolution we obtain the empty clause, and hence proof $c$
Resolution

- Resolution is a single inference rule
- It takes two clauses, and produces one new clause
- The new clause is implied by the two old clauses
  - The two old clauses need to have complementary literals
  - The new clause contains all the literals of both old clauses except the complementary ones
- Terminates when the empty clause is produced, i.e., a proof has been found
Resolution example

[c, ¬a, ¬b] [a] [b] [¬c]
Resolution example

[c, ¬a, ¬b] [a] [b] [¬c]
Resolution example

\[
\begin{align*}
[c, \neg a, \neg b] & \quad [a] & \quad [b] & \quad [\neg c] \\
[c, \neg b] & \quad [b] & \quad [\neg c]
\end{align*}
\]
Resolution example

[c,¬a,¬b] [a] [b] [¬c]

[c,¬b] [b] [¬c]
Resolution example

\[
\begin{align*}
&[c, \neg a, \neg b] \quad [a] \quad [b] \quad [\neg c] \\
&[c, \neg b] \quad [b] \quad [\neg c] \\
&[c] \quad [\neg c]
\end{align*}
\]
Resolution example

\[[c, \neg a, \neg b] \quad [a] \quad [b] \quad [\neg c]\]

\[[c, \neg b] \quad [b] \quad [\neg c]\]

\[[c] \quad [\neg c]\]
Resolution example
Why Horn clauses?

• Resolution of two Horn clauses always results in a Horn clause
• Resolution of a goal clause and a definite clause is always a goal clause
• Horn clauses have better computational properties than normal clauses
• Prolog is based on computing with Horn clauses
Alfred Horn

• The name *Horn clause* comes from Alfred Horn, who discovered the significance of such clauses.